Engineering Notes

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Trilateration and Extension to Global Positioning System Navigation

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Introduction

TRILATERATION is the determination of a navigation position based on simultaneous ranging from three trackers. It is a problem of simple geometry and a solution algorithm is given by Escobal.[1] The Escobal algorithm is unnecessarily complex. By referencing the navigation position to the plane defined by the three trackers, simple explicit solutions are obtained in this Note. The trilateration problem is, of course, trivial. However, it will be shown that the method can be easily extended to Global Positioning System (GPS) navigation,[2] which requires the determination of time in addition to the position.

Trilateration

The geometry of trilateration is shown in Fig. 1. The $\boldsymbol{b}$ and the $\boldsymbol{r}$ represent the tracker baseline vectors and the ranging vectors, respectively. Obviously,

$$r_2^2 = \boldsymbol{r}_2 * \boldsymbol{r}_2 = (\boldsymbol{r}_1 - \boldsymbol{b}_2)*(\boldsymbol{r}_1 - \boldsymbol{b}_2) = r_1^2 + b_2^2 - 2\boldsymbol{r}_1 * \boldsymbol{b}_2 \tag{1}$$

Similarly

$$r_3^2 = r_1^2 + b_3^2 - 2\boldsymbol{r}_1 * \boldsymbol{b}_3 \tag{2}$$

These equations give the projections of the unknown position vector $\boldsymbol{r}_1$ along the known tracking baselines $\boldsymbol{b}_2$ and $\boldsymbol{b}_3$ in terms of the measured ranges r_1, r_2, and r_3. Since the tracking baselines are not necessarily orthogonal, it is convenient to construct a set of orthogonal unit vectors from $\boldsymbol{b}_2$ and $\boldsymbol{b}_3$ as follows:

$$\boldsymbol{i} = \boldsymbol{b}_2 / b_2 \tag{3}$$

$$\boldsymbol{j} = \frac{\boldsymbol{b}_3 - (\boldsymbol{b}_3 * \boldsymbol{i})\boldsymbol{i}}{|\boldsymbol{b}_3 - (\boldsymbol{b}_3 * \boldsymbol{i})\boldsymbol{i}|} \tag{4}$$

$$\boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j} \tag{5}$$

The components of $\boldsymbol{r}_1$ along $\boldsymbol{i}$ and $\boldsymbol{j}$ may now be obtained from Eqs. (1) and (2) as

$$X = \boldsymbol{r}_1 * \boldsymbol{i} = (r_1^2 - r_2^2 + b_2^2)/2b_2 \tag{6}$$

$$Y = \boldsymbol{r}_1 * \boldsymbol{j} = \frac{r_1^2 - r_3^2 + b_3^2 - 2(\boldsymbol{b}_3 * \boldsymbol{i})X}{2(\boldsymbol{b}_3 * \boldsymbol{j})} \tag{7}$$

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Equations (6) and (7) define the projection of the navigation position in the 1-2-3 tracker plane. The height above the plane is

$$Z = \boldsymbol{r}_1 * \boldsymbol{k} = \pm\sqrt{r_1^2 - X^2 - Y^2} \tag{8}$$

Since measurements contain errors, under unusual circumstances (for instance, when the navigation position is in the tracker plane and all three range measurements are short), the expression under the square root sign may become negative and trilateration has no solution. Generally, however, the ± signs in Eq. (8) represent two possible solutions that are mirror images with respect to the tracker plane. The existence of dual solutions is inherent to trilateration. The choice of a solution cannot be resolved from algebra as claimed in Ref. 1, but must be determined from other information. For instance, if the trackers are terrestrial tracking stations, one solution may locate the navigation position inside of the Earth and is an erroneous solution. On the other hand, the other solution is erroneous, if the trackers are surface ships and the tracked vehicle is a submarine.

If the tracking baselines happen to be collinear, the tracker plane is undefined. Equation (7) degenerates to the following equation similar to, but generally inconsistent with Eq. (6):

$$X = (r_1^2 - r_3^2 + b_3^2)/2b_3 \tag{9}$$

For rare combinations of measurement errors, Eqs. (6) and (9) can become consistent. In that case, the navigation position lies on a circle centered on the baseline at a distance X from tracker 1 with a radius equal to $\sqrt{r_1^2 - X^2}$.

Global Positioning System Navigation

The simplicity of our trilateration solution and the physical insight it provides result from the choice of the plane of the trackers as a reference. The method can be extended to navigation using GPS data. GPS navigation differs from trilateration in that ranging based on one-way radio signal transit time contains a bias equal to the navigation user clock error t multiplied by the speed of light C. For this reason, the range measurements are called pseudoranges; pseudoranges from four trackers (12-h period Earth satellites) are required to compute navigation position and time.

Let the r represent ranges as before, but let R1, R2, R3, and R4 represent pseudoranges to the four trackers. Equations (6-8) remain valid, although the ranges differ from the measured pseudoranges by an unknown common bias, i.e.,

$$R_n - Ct = r_n > 0 \qquad (n = 1,2,3,4) \tag{10}$$

From this relation, one obtains

$$r_n - r_m = R_n - R_m$$

$$r_n^2 - r_m^2 = R_n^2 - R_m^2 + 2(R_m - R_n)Ct$$

$$(m = 1,2,3,4)$$

Therefore, Eqs. (6) and (7) become

$$X = \frac{R_1^2 - R_2^2 + b_2^2 + 2(R_2 - R_1)Ct}{2b_2} \tag{11}$$

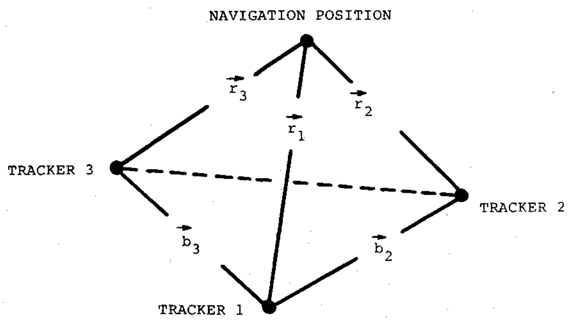

Fig. 1 Trilateration geometry.

$$Y=\frac{R_1^2-R_3^2+b_3^2+2(R_3-R_1)Ct-2(\boldsymbol{b}_3{}^*\boldsymbol{i})X}{2(\boldsymbol{b}_3{}^*\boldsymbol{j})} \tag{12}$$

Notice that the position components are linear functions of the clock error. By eliminating the clock error from these equations one obtains

$$\begin{aligned}&2[b_2(R_3-R_1)-(\boldsymbol{b}_3{}^*\boldsymbol{i})(R_2-R_1)]X\\&-2(\boldsymbol{b}_3{}^*\boldsymbol{j})(R_2-R_1)Y\\&+(R_2-R_1)(R_1^2-R_3^2+b_3^2)\\&-(R_3-R_1)(R_1^2-R_2^2+b_2^2)=0\end{aligned} \tag{13}$$

which defines a line in the 1-2-3 tracker plane instead of a point, as in the case of trilateration without clock error. So far the pseudorange information from the fourth tracker has not been made use of. From the fourth pseudorange R_4 and following a procedure similar to that used in deriving Eqs. (11) and (12), one obtains

$$\begin{aligned}Z=\boldsymbol{r}_1{}^*\boldsymbol{k}=[&R_1^2-R_4^2+b_4^2+2(R_4-R_1)Ct-2(\boldsymbol{b}_4{}^*\boldsymbol{i})X\\&-2(\boldsymbol{b}_4{}^*\boldsymbol{j})Y]/2(\boldsymbol{b}_4{}^*\boldsymbol{k})\end{aligned} \tag{14}$$

Equations (11), (12), (14), and the following additional equation

$$(R_1-Ct)^2=r_1^2=X^2+Y^2+Z^2 \tag{15}$$

constitute four simulataneous equations for the three navigation position coordinates and the clock error. Since the position coordinates are linear functions of the clock error as given in Eqs. (11), (12), and (14), substitution of these expressions into Eq. (15) results in the following quadratic equation for the clock error:

$$\begin{aligned}f(Ct)&=A^*(Ct)^2-2B^*(Ct)+D\\&=(R_1-Ct)^2-X^2-Y^2-Z^2=0\end{aligned} \tag{16}$$

with solution

$$Ct=(B\pm\sqrt{B^2-A^*D})/A \tag{17}$$

The coefficients A, B, and D are expressible as

$$D=f(0)=R_1^2-X_0^2-Y_0^2-Z_0^2 \tag{18}$$

$$B=-\tfrac{1}{2}f'(0)=R_1+X_0X_0'+Y_0Y_0'+Z_0Z_0' \tag{19}$$

$$A=\tfrac{1}{2}f''(0)=1-X_0'^2-Y_0'^2-Z_0'^2 \tag{20}$$

where ($'$) denotes differentiation and X_0, Y_0, Z_0, X_0', Y_0', and Z_0' are defined below and obtainable from Eqs. (11), (12), and (14) as

$$X_0=X_{Ct=0}=\frac{R_1^2-R_2^2+b_2^2}{2b_2}$$

$$Y_0=Y_{Ct=0}=\frac{R_1^2-R_3^2+b_3^2-2(\boldsymbol{b}_3{}^*\boldsymbol{i})X_0}{2(\boldsymbol{b}_3{}^*\boldsymbol{j})}$$

$$Z_0=Z_{Ct=0}=\frac{R_1^2-R_4^2+b_4^2-2(\boldsymbol{b}_4{}^*\boldsymbol{i})X_0-2(\boldsymbol{b}_4{}^*\boldsymbol{j})Y_0}{2(\boldsymbol{b}_4{}^*\boldsymbol{k})}$$

$$X_0'=X_{Ct=0}'=\frac{R_2-R_1}{b_2}$$

$$Y_0'=Y_{Ct=0}'=\frac{(R_3-R_1)-(\boldsymbol{b}_3{}^*\boldsymbol{i})X_0'}{(\boldsymbol{b}_3{}^*\boldsymbol{j})}$$

$$Z_0'=Z_{Ct=0}'=\frac{(R_4-R_1)-(\boldsymbol{b}_4{}^*\boldsymbol{i})X_0'-(\boldsymbol{b}_4{}^*\boldsymbol{j})Y_0'}{\boldsymbol{b}_4{}^*\boldsymbol{k}}$$

When the clock error is known, nvigation position components follow immediately from Eqs. (11), (12), and (14).

In deriving the equation for the clock error, the constraints given by the inequality in Eq. (10), $R_n-Ct>0$, are not used. In the presence of measurement errors, there is no guarantee that at least one of the solutions given by Eq. (17) satisfies the constraints and is a valid solution, although one would expect that, in general, a valid solution exists if measurement errors are small. The solutions given by Eq. (17) must be tested against the inequality and, depending on the situation, one, two, or no solutions exist. Obviously no solution exists if the clock errors given by Eq. (17) are complex. Numerical results show that, even if they are both real, a solution may not exist if arbitary measurement errors are allowed.

Another special situation exists when the four trackers happen to be coplanar. In that case Eq. (14) degenerates to

$$R_1^2-R_4^2+b_4^2+2(R_4-R_1)Ct-2(\boldsymbol{b}_4{}^*\boldsymbol{i})X-2(\boldsymbol{b}_4{}^*\boldsymbol{j})Y=0$$

This equation, coupled with Eqs. (11) and (12), is generally sufficient to solve for the clock error Ct and the two position coordinates X and Y. The height Z then follows from Eq. (8). As in the case of trilateration, two solutions exist.

Although the standard measurements provided by GPS are pseudoranges, it is also possible to obtain pseudorange rate information from Doppler shifts of the carrier signal.[3] The pseudorange-rate signal can be written as

$$\dot{R}_n-C\dot{t}=(\boldsymbol{v}-\boldsymbol{v}_n)^*\boldsymbol{r}_n/|\boldsymbol{r}_n| \quad (n=1,2,3,4) \tag{21}$$

where $\boldsymbol{v}$ is the unknown velocity vector of the navigation vehicle, $\dot{t}$ the unknown clock drift rate, $\boldsymbol{v}_n$ the known velocity vector of tracker n, and $\dot{R}_n$ is the measured pseudorange-rate.

When the navigation position is known from the method described above, Eq. (21) represents four simultaneous linear equations and can be solved immediately for the three velocity components and the clock drift rate.

Discussion

The reduction of the GPS navigation problem to the solution of a quadratic equation was first described by Bancroft.[4] The advantage of the method over a four-dimensional iterative solution algorithm is obvious, particularly when a good initial guess to start the iteration is not available. The Bancroft algorithm requires the inversion of a 4×4 matrix. The present algorithm is simpler and also provides better physical insights to the problem.

References

[1]Escobal, P.R., *Methods of Orbit Determination,* John Wiley & Sons, New York, 1965, pp. 309-311.

[2]Jorgensen, P.S., "Navstart/Global Positioning System 18-Satellite Constellations" *Global Positioning System,* Vol. 2, The Institute of Navigation, Washington, DC, 1984, pp. 2-12.

[3]Fang, B.T. and Seifert E., "An Evaluation of Global Positioning System Data for Landsat-4 Orbit Determination," AIAA Paper 85-0268, Jan. 1985.

[4]Bancroft, S., "An Algebraic Solution of the GPS Equation," *IEEE Transactions on Aerospace and Electronics Systems,* Vol. AES-21, Jan. 1985. pp. 56-59.

Auxiliary Problem Concerning Optimal Pursuit on Lagrangian Orbits

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Introduction

A NUMBER of particular cases of pursuit problems have been studied recently. Rozenberg[1] considers a plane pursuit by assuming that the angular velocity of the pursuer is bounded. Simakov[2] determines the rendezvous time when the displacement velocity of the two vehicles and the slopes of the trajectories are subject to constraints. Burrow and Rishel[3] have studied the characteristics of time-optimal trajectories when the acceleration and angular velocity are limited, but the direction of the interception remains free. Reference 4 establishes the conditions necessary for rendezvous in the case of constant velocity when the radius of curvature of the trajectory is subject to a constraint. The studies reported in Refs. 5 and 6 concern the pursuit on Lagrangian orbits of the Earth-moon system assuming that the acceleration is limited.

The present Note tackles time-optimal pursuit by using an auxiliary problem concerning the distance between the two vehicles. The performance index of the auxiliary problem is the value of the distance at the moment when the trajectory reaches the terminal surface S, considered as a convex closed set. We must mention that the derivative of the distance with respect to time does not depend explicitly on the components of the acceleration taken as controls. Thus, the domain of the states where the motion is analyzed is obtained. For the sake of a simple treatment, we take a change of function, defining the difference between the state variables of the pursuer and pursued vehicles.

Formulation of Problem

Let us consider a rotating system of axes with the origin at one of the Lagrangian colinear points of the Earth-moon system. The equations of motion of a space vehicle acted upon by a small propulsion force that moves in orbit around these points have been given in Ref. 7. The motion of two vehicles is governed by the equations

$$\frac{dx_1^j}{dt}=x_2^j, \qquad \frac{dx_2^j}{dt}=k_1x_1^j+2\omega x_4^j+u_1^j$$

$$\frac{dx_3^j}{dt}=x_4^j, \qquad \frac{dx_4^j}{dt}=-2\omega x_2^j+k_2x_3^j+u_2^j \quad (j=1,2) \tag{1}$$

where $X^j(x_1^j,\ldots,x_4^j)$ is a point in the physical space R_4 and $U^j=(u_1^j,\ u_2^j)$ represents the closed domain of the controls, which satisfy the conditions $|u_k^j|\le\bar{u}_k^j$ $(k=1,2)$. The indices 1 and 2 stand for the pursuing and the pursued vehicles, respectively. We note that the components of the velocities and accelerations of the pursuing vehicle are greater than those of the pursued vehicle. Let us put $x_i=x_i^1=x_i^2$ $(i=1,\ldots,4)$. Our aim is to determine the trajectory generated by the controlled system,

$$\frac{dx_1}{dt}=x_2, \qquad \frac{dx_2}{dt}=k_1x_1+2\omega x_4+u_1^1-u_1^2$$

$$\frac{dx_3}{dt}=x_4, \qquad \frac{dx_4}{dt}=-2\omega x_2+k_2x_3+u_2^1-u_2^2 \tag{2}$$

such that the distance between the two vehicles, given by $R_2^2(x)=x_1^2+x_3^2$, is minimal on the terminal surface.

Auxiliary Problem

Assume that the distance between the two vehicles $R_2[x(t)]$ is minimal at $t=t^*$. Then, $\dot{R}_2[x(t^*)]=0$. The system of the equations of motion for the pursuit problem may be written as

$$\frac{dx}{dt}=f(x,u^1,u^2), \quad u^1\in U^1, \quad u^2\in U^2 \tag{3}$$

where x is the four-dimensional vector of the physical space and u^1, u^2 control vectors subject to constraints similar to $u^j\in U^j$. Let us consider the derivative by virtue of Eq. 3 of the expression $\frac{1}{2}R^2[x(t)]$. We have $A=(d/dt)[R_2^2(x)/2]$.

The function $R^2(x,t)$ decreases monotonically with respect to time on each pursuit trajectory. Thus, the domain containing the set of all states possessing this property is $D=\{x;A(x)<0\}$. The terminal surface coincides with the boundary $S=\{x;\dot{R}^2(x)=0\}$ of the domain D. The minimal value of the function $R^2(x)$, denoted by $V(x)$, is attained on the terminal surface S. The pursuer tends to attain the minimal value $V(x)$ and the pursued vehicle tends to maximize this value. Thus, we are led to satisfy the equation

$$\operatorname*{val}_{u^1,u^2}\ [\operatorname{grad}V(x),f(x,u^1,u^2)]=0 \tag{4}$$

with the boundary condition on the terminal surface

$$V(x)_{x\in S}=R^2(x) \tag{5}$$

The solution satisfying Eqs. (4) and (5) determines the performance index $V(x)$ only when the trajectory corresponding to this solution permits attaining the terminal surface for every trajectory of the pursued vehicle. The optimal pursuit implies the determination of the time necessary to attain the surface S when the results of the problem of the distance are used. We call this the auxiliary problem.

Optimal Pursuit

Let T be the time necessary for the pursuit trajectory to attain the surface S. The domain of admissible states of the auxiliary problem becomes $D=\{x;x_1x_2+x_3x_4<0\}$. The terminal surface may be expressed in parametric form

$$S=\{x;x_i=s_i\,(i=1,2,3), \qquad s_1s_2+s_3x_4(s)=0\}$$

As the performance index of the auxiliary problem, we take

$$V(x)\,|_s=\tfrac{1}{2}(s_1^2+s_3^2) \tag{6}$$

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